

FARRELL-JONES CONJECTURE FOR THE SOLVABLE BAUMSLAG-SOLITAR GROUPS

TOM FARRELL AND XIAOLEI WU

ABSTRACT. In this paper, we prove the Farrell-Jones Conjecture for the solvable Baumslag-Solitar groups with coefficients in an additive category. We also extend our results to groups of the form, $\mathbb{Z}[1/p]$ semidirect product with any virtually cyclic group, where p is a prime number.

1. INTRODUCTION

In the Farrell and Linnell paper [13], they proved that if the fibered isomorphism conjecture is true for all nearly crystallographic groups, then it is true for all virtually solvable groups. However, they were not able to verify the fibered isomorphism conjecture for all nearly crystallographic groups. In particular, they pointed out that the fibered isomorphism conjecture has not been verified for the group $\mathbb{Z}[\frac{1}{2}] \rtimes_{\alpha} \mathbb{Z}$, where α is multiplication by 2. Note this group is isomorphic to the Baumslag-Solitar group $BS(1, 2)$. Recall that the Baumslag-Solitar group $BS(m, n)$ is defined by $\langle a, b | ba^mb^{-1} = a^n \rangle$ and all the solvable ones are isomorphic to $BS(1, d)$. Note that $BS(m, n) \cong BS(n, m) \cong BS(-m, -n)$. Using new technology developed by Bartels, Lück and Reich in [3], [4], [5], [6], we prove the following result.

Theorem 1.1. *The K-theoretic and L-theoretic Farrell-Jones Conjecture is true for the solvable Baumslag-Solitar groups with coefficients in an additive category.*

Note the truth of the Farrell-Jones conjecture with coefficients in an additive category implies the fibered isomorphism conjecture. For more information about Farrell-Jones conjecture and its fibered version, see for example [12]. For precise formulation and discussion of Farrell-Jones conjecture with coefficients in an additive category, see for example [2], [6]. We will first prove the Farrell-Jones

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conjecture for $BS(1, d)$, where $d > 1$. The $d < -1$ case will be taken care of in section 6; cf. Corollary 6.1. Note that $BS(1, 1)$ and $B(1, -1)$ are the fundamental groups of the torus and Klein bottle respectively. And the Farrell-Jones conjecture is known for those groups. The authors want to point out here that our current method can not be applied to all Baumslag-Solitar groups. For example, we do not know whether the Farrell-Jones conjecture is true for the group $BS(2, 3)$. For convenience we will denote the group $BS(1, d)$ as $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$ for the rest of the paper.

Our strategy is to show that $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$ is in fact a Farrell-Hsiang group, defined by Bartels and Lück in [3]. The main difficulty is to find a suitable flow space, ours is a modification of Bartels and Lück's flow space in [4]. Once our flow space is defined, we prove in Lemma 3.3 that it has a lot of good properties, which implies that it has a \mathcal{VCyc} -cover by results of Bartels, Lück and Reich in [5]. It should be mentioned that our results rely heavily on the existence of such a cover. In the proof of our main theorem, we also use a lemma of Bartels, Lück and Reich from [6], proposition 5.3. In the last section, we extend our current result to any group of the form $\mathbb{Z}[\frac{1}{p}] \rtimes C$, where p is any prime number, C is any virtually cyclic group.

In this paper, without further assumption, when we write $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$, α is always multiplication by d , and we will assume d is a fixed integer bigger than 1. When we write Farrell-Jones conjecture, we mean the Farrell-Jones conjecture with coefficients in an additive category. Let G be a (discrete) group acting on a space X . We say the action is proper if for any $x \in X$ there is an open neighborhood U of x such that $\{g \in G \mid gU \cap U \neq \emptyset\}$ is finite.

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2. A MODEL FOR $E(\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z})$

In this section, we give a model for $E(\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z})$, a contractible space with free, proper and discontinuous Γ action, where Γ is the group $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$. We also put a metric on it, such that Γ acts isometrically on $E(\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z})$. Most part of the material in this section has been well studied before, see for example [10], section 7.4, and [14].

Let X be $S^1 \times [0, 1] / (z, 0) \sim (z^d, 1)$. Its fundamental group is $\{x, k \mid kxk^{-1} = x^d\}$, which is isomorphic to our group $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$. Since its universal cover is contractible, it is a model for $E(\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z})$.

Let T_d be the oriented infinite, regular, $(d + 1)$ -valent tree with edge length 1 (see Figure 1). At each vertex there are d incoming and 1 outgoing edges. It is not too hard to show that the universal cover of X is $T_d \times \mathbb{R}$. However, it is not easy to figure out how Γ acts on $T_d \times \mathbb{R}$. So what we do here is give an action of Γ on $T_d \times \mathbb{R}$, and show that $T_d \times \mathbb{R} / \Gamma$ is homeomorphic to X .

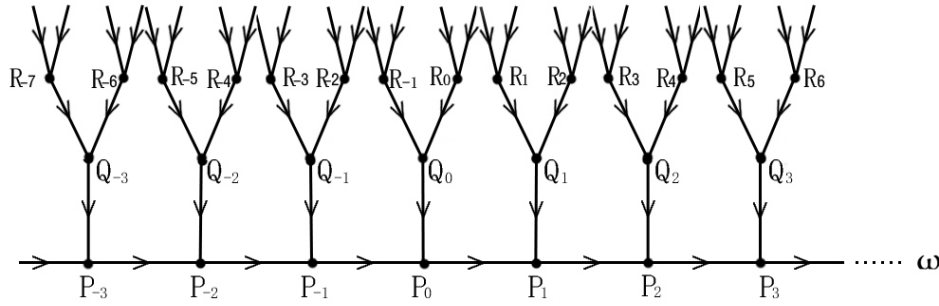


FIGURE 1. T_2 : Oriented Infinite Regular 3-valent Tree

We first assume d is a prime number. We can embed Γ into $GL_2(\mathbb{Q})$ by mapping $(x, k) \in \mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$ to $\begin{pmatrix} d^{-k} & x \\ 0 & 1 \end{pmatrix}$. There is a natural action of $GL_2(\mathbb{Q})$ on this tree. We explain briefly how Γ acts on it. For more information, see for example [16], page 69 - 78 or [17]. Note that our group is an HNN extension, hence we can also give an action of Γ on the tree by Bass-Serre theory. It turns out that they are the same action.

Note that every element $\begin{pmatrix} d^{-k} & x \\ 0 & 1 \end{pmatrix}$ in Γ can be written as $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-k} & 0 \\ 0 & 1 \end{pmatrix}$, so we only need to show how these two types of elements act on T_d . Choose an oriented line L_0 inside the tree T_d and a vertex P_0 in L_0 as the basepoint. Denote the infinite point along L_0 towards the positive direction as ω ; cf. Figure 1. For any $z \in T_d$, there is a unique geodesic starting from z and moving towards ω ; denote it as $[z, \omega)$. Note there is no canonical choice of L_0 or P_0 , however, ω is independent of the choice of L_0 or P_0 . Define the action of $\begin{pmatrix} d^{-k} & 0 \\ 0 & 1 \end{pmatrix}$ on the tree by translation along L_0 , moving along the line L_0 towards the positive direction by k units. For example in the case $d = 2$ as in Figure 1, we choose $\dots P_{-2}P_{-1}P_0P_1P_2\dots$ as L_0

and $\begin{pmatrix} d^{-k} & 0 \\ 0 & 1 \end{pmatrix}$ acts on the tree by translation along L_0 , moving right along the line L_0 by k units. Hence P_n will be translated to P_{n+k} , and every vertex or edge growing on the tree with root P_n will also be translated to the corresponding vertex or edge growing from root P_{n+k} . Note that any $x \in \mathbb{Z}[\frac{1}{d}]$ can be written uniquely as the form $d^{-m}s$, where $(s, d) = 1$, and $\begin{pmatrix} 1 & d^{-m}s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d^{-m} \\ 0 & 1 \end{pmatrix}^s$, hence we only need to explain how $\beta_m = \begin{pmatrix} 1 & d^{-m} \\ 0 & 1 \end{pmatrix}$ acts on T_d . First we give some notation and terminology. We define a Busemann function $f_d : T_d \rightarrow \mathbb{R}$, by mapping the line L_0 isometrically to \mathbb{R} with P_0 mapped to 0 and ω to $-\infty$ (orientation reversed); for an arbitrary point Q , define $f_d(Q) = f_d(P) + d(P, Q)$, where P is the closest point in the line L_0 from Q . For example in the case $d = 2$ as in Figure 1, we map L_0 to the real line \mathbb{R} with $f_d(P_n) = -n$. Then for example $f_d(R_1) = 1$ since the closest point to R_1 is P_1 and $d(R_1, P_1) = 2$. Let $H_n = f_d^{-1}(n)$ which we call a horosphere in T_d with center ω . Likewise, let $B_n = f_d^{-1}((-\infty, n])$ be the corresponding horoball; cf. [1], p23] for this terminology. And if $z \in H_n$, let $T_d(z)$ be the subtree in T_d rooted at z and growing outside B_n . Finally, for each $z \in H_n$ and $l \in \mathbb{Z}^+$, let

$$S(z, l) = \{z' \in T_d(z) \mid d(z, z') = l\}.$$

Remark 2.1. *These definitions of f_d , H_n , B_n and $S(z, l)$ for the tree T_d are valid for any positive integer d (not just for primes).*

With the terminology above, we describe the action of β_m on T_d as follows:

- (i). β_m fixes each point in B_{-m}
- (ii). For each $z \in H_{-m}$ and $l \in \mathbb{Z}^+$, β_m leaves $S(z, l)$ invariant and cyclically permutes its members.

Note that P_n has isotropy $\{\begin{pmatrix} 1 & d^{-n}m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z}\}$. One of the good things about this action is that it fixes the infinite point ω . We also define the action of Γ on \mathbb{R} by $\begin{pmatrix} d^{-k} & x \\ 0 & 1 \end{pmatrix} w = d^{-k}w + x$, for $w \in \mathbb{R}$.

With these preparations, we define the action of Γ on $T_d \times \mathbb{R}$ to be the diagonal action, i.e. $g(z, w) = (gz, gw)$, for $g \in \Gamma$, $(z, w) \in T_d \times \mathbb{R}$. It is not hard to check that Γ acts freely, properly and discontinuously on $T_d \times \mathbb{R}$. A fundamental domain is $P_0P_1 \times [0, 1]$, while very luckily $T_d \times \mathbb{R}/\Gamma$ is homeomorphic to our previous space X . So $T_d \times \mathbb{R}$ is a model for $E(\mathbb{Z}[\frac{1}{d}] \rtimes_a \mathbb{Z})$.

Remark 2.2. Let G_d be the group $\left\{ \begin{pmatrix} d^n \cdot \frac{s_1}{s_2} & b \\ 0 & 1 \end{pmatrix} \mid s_1, s_2 \text{ are positive integers coprime to } d, b \in \mathbb{Q} \right\}$. Note Γ is a subgroup of G_d and the action of Γ on T_d can be extended to G_d for any $d > 1$ which will be explained in the Appendix. On the other hand, our definition of the Γ action on \mathbb{R} can also be easily extended to G_d ; in fact, for any $w \in \mathbb{R}$, $\begin{pmatrix} d^n \cdot \frac{s_1}{s_2} & b \\ 0 & 1 \end{pmatrix}$ acts on \mathbb{R} by $\begin{pmatrix} d^n \cdot \frac{s_1}{s_2} & b \\ 0 & 1 \end{pmatrix} w = d^n \cdot \frac{s_1}{s_2} \cdot w + b$. Hence we can define an action of G_d on $T_d \times \mathbb{R}$ by the diagonal action for any $d > 1$. Furthermore, Γ as a subgroup of G_d acts on $T_d \times \mathbb{R}$ and $T_d \times \mathbb{R}/\Gamma$ is homeomorphic to our previous space X . Note that this action is the "boundary" of the natural (isometric) diagonal action of G_d on $T_d \times \mathbb{H}^2$ in the upper half plane model for \mathbb{H}^2 where we identify \mathbb{R} in $T_d \times \mathbb{R}$ with the x -axis of \mathbb{R}^2 . Moreover, elements of the form $\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ fix the line L_0 in T_d and do some permutation on each tree growing on L_0 where s is an positive integer coprime to d . One important observation is that the action of $g = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ on T_d does not change the value of the Busemann function; i.e., $f_d(g(z, w)) = f_d(x, w)$, for every $(z, w) \in T_d \times \mathbb{R}$.

We now define a metric on $T_d \times \mathbb{R}$ so that Γ acts isometrically on it. Note first that both the tree T_d and the real line \mathbb{R} already have canonical metrics. Now we define the metric on $T_d \times \mathbb{R}$ by the warped product of T_d and \mathbb{R} with respect to the warping function d^{-f_d} , where f_d is the Busemann function we defined before. $T_d \times \mathbb{R}$ is a metric space under this metric. For more information about warped product on metric spaces, see for example [9]. If we restricted to the two dimensional subspace $L_0 \times \mathbb{R}$, the metric will be

$$dz^2 + (d^{-f_d(z)})^2 dw^2, \text{ where } z \in L_0, w \in \mathbb{R}.$$

Simple calculations show that this is a space with constant curvature $-(\ln d)^2$. Hence $T_d \times \mathbb{R}$ is constructed by gluing together infinitely many copies of hyperbolic planes with curvature $-(\ln d)^2$ along $[z, \omega) \times \mathbb{R}$, where z can be any vertex in T_d . In fact, each line L in T_d going towards ω determined a hyperbolic plane $H_L = L \times \mathbb{R}$. And if $z \in T_d$ is the first point L and L' meet towards ω , then H_L and $H_{L'}$ are glued together along $[z, \omega) \times \mathbb{R}$. Note if we identify $L \times \mathbb{R}$ with the Poincaré disk model (with curvature $-(\ln d)^2$), $z \times \mathbb{R}$ will be mapped to a horosphere. Correspondingly, $[z, \omega)$ will be mapped to the region outside the horoball (the disk bounded by the horosphere) in the Poincaré disk, which is not convex. Hence, $T_d \times \mathbb{R}$ might not

be a CAT(0) space, in fact, our group is not a CAT(0) group since it contains a non-finitely generated Abelian subgroup; cf. Corollary 7.6 in [8], p247.

We are going to need the following lemmas in the future.

Lemma 2.1. *For any two points $(z_1, w_1), (z_2, w_2) \in T_d \times \mathbb{R}$,*

$$d_{T_d \times \mathbb{R}}((z_1, w_1), (z_2, w_2)) \geq d_{T_d \times \mathbb{R}}((z_1, w_2), (z_2, w_2)) = d_T(z_1, z_2)$$

Proof See for example [9], Lemma 3.1, Lemma 3.2. \square

Remark 2.3. *It is not always true that $d_{T_d \times \mathbb{R}}((z_1, w_1), (z_2, w_2)) \geq |w_1 - w_2|$.*

Corollary 2.1. *Let $z_1, z_2 \in T_d$, $w_1, w_2 \in \mathbb{R}$, then the following inequality holds*

$$d_{T_d \times \mathbb{R}}((z_1, w_1), (z_2, w_2)) \geq \frac{1}{2} d_{T_d \times \mathbb{R}}((z_1, w_1), (z_1, w_2))$$

Proof By the triangle inequality on metric spaces, we have

$$d_{T_d \times \mathbb{R}}((z_1, w_1), (z_2, w_2)) + d_{T_d \times \mathbb{R}}((z_2, w_2), (z_1, w_2)) \geq d_{T_d \times \mathbb{R}}((z_1, w_1), (z_1, w_2)).$$

By Lemma 2.1, $d_{T_d \times \mathbb{R}}((z_1, w_1), (z_2, w_2)) \geq d_T(z_1, z_2) = d_{T_d \times \mathbb{R}}((z_2, w_2), (z_1, w_2))$, therefore $2d_{T_d \times \mathbb{R}}((z_1, w_1), (z_2, w_2)) \geq d_{T_d \times \mathbb{R}}((z_1, w_1), (z_1, w_2))$. \square

Lemma 2.2. *Let z_0 be a fixed point in T_d , w_1, w_2, w_3 are three different points in \mathbb{R} such that $|w_1 - w_2| < |w_1 - w_3|$, then*

$$d_{T_d \times \mathbb{R}}((z_0, w_1), (z_0, w_2)) < d_{T_d \times \mathbb{R}}((z_0, w_1), (z_0, w_3))$$

Proof It is not hard to see that we can arrange the geodesics connecting (z_0, w_1) and (z_0, w_i) , $i=2,3$, to lie within a single hyperbolic plane. Hence we only need to prove the lemma in the hyperbolic plane $L_0 \times \mathbb{R}$. We can map $L_0 \times \mathbb{R}$ to the Poincaré disk model with (z_0, w_1) as the center; $\{z_0\} \times \mathbb{R}$ will be mapped to a horosphere. Now our lemma follows easily. \square

Lemma 2.3. *Let z_0 be a fixed point in T_d , w_1, w_2 are two fixed points in \mathbb{R} , denote the distance $d_{T_d \times \mathbb{R}}((z_0, \frac{w_1}{n}), (z_0, \frac{w_2}{n}))$ by D_n , then for any given integer $n > 0$,*

$$\sinh\left(\frac{\ln d}{2} D_1\right) = n \sinh\left(\frac{\ln d}{2} D_n\right)$$

Therefore,

$$D_n = \frac{2}{\ln d} \operatorname{arcsinh}\left(\frac{1}{n} \sinh\left(\frac{\ln d}{2} D_1\right)\right)$$

In particular,

$$\lim_{n \rightarrow \infty} D_n = 0.$$

Proof Denote the induced inner metric on $\{z_0\} \times \mathbb{R}$ as $d_{\mathbb{R}}$, then $d_{\mathbb{R}}(w_1, w_2) = n d_{\mathbb{R}}(\frac{w_1}{n}, \frac{w_2}{n})$. Note if we use the Poincaré disk model, $\{z_0\} \times \mathbb{R}$ will be mapped to a horosphere, hence for any two points $w_1, w_2 \in \{z_0\} \times \mathbb{R}$,

$$d_{\mathbb{R}}(w_1, w_2) = \frac{2}{\ln d} \sinh\left(\frac{\ln d}{2} d_{T_d \times \mathbb{R}}((z_0, w_1), (z_0, w_2))\right)$$

see for example [11], Theorem 4.6. Now apply this to both sides of the equation $d_{\mathbb{R}}(w_1, w_2) = n d_{\mathbb{R}}(\frac{w_1}{n}, \frac{w_2}{n})$, our lemma follows. \square

3. FLOW SPACE FOR $T_d \times \mathbb{R}$

In this section we define a flow space for $E(\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z})$, which is a modification of the flow space defined in Bartels and Lück's paper [4]. Then we prove that it has lots of good properties, which guarantee that it has a long thin cover.

We first introduce Bartels and Lück's flow space starting with the notion of generalized geodesic.

Definition 3.1. Let X be a metric space. A continuous map $c : \mathbb{R} \rightarrow X$ is called a **generalized geodesic** if there are $c_-, c_+ \in \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ satisfying

$$c_- \leq c_+, c_- \neq \infty, c_+ \neq -\infty$$

such that c is locally constant on the complement of the interval $I_c := (c_-, c_+)$ and restricts to an isometry on I_c .

Definition 3.2. Let (X, d_X) be a metric space. Let $FS(X)$ be the set of all generalized geodesics in X . We define a metric on $FS(X)$ by

$$d_{FS(X)}(c, d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}} dt$$

The flow on $FS(X)$ is defined by

$$\Phi : FS(X) \times \mathbb{R} \rightarrow FS(X)$$

where $\Phi_{\tau}(c)(t) = c(t + \tau)$ for $\tau \in \mathbb{R}$, $c \in FS(X)$ and $t \in \mathbb{R}$.

Lemma 3.1. The map Φ is a continuous flow and if we let $c, d \in FS(X)$, $\tau \in \mathbb{R}$, then the following inequality holds

$$e^{-|\tau|} d_{FS(X)}(c, d) \leq d_{FS(X)}(\Phi_{\tau}(c), \Phi_{\tau}(d)) \leq e^{|\tau|} d_{FS(X)}(c, d)$$

Proof We leave the proof to the reader. A more general version is proved in A. Bartels and W. Lück's paper [4], Lemma 1.3. \square

Note that the isometry group of (X, d_X) acts canonically on $FS(X)$. Recall a map is proper if the inverse image of every compact subset is compact. Bartels and Lück also proved the following for the flow space $FS(X)$ in [4].

Proposition 3.1. *If (X, d_X) is a proper metric space, then $(FS(X), d_{FS(X)})$ is a proper metric space, in particular it is a complete metric space. Furthermore, if a group Γ acts isometrically and properly on (X, d_X) , then Γ also acts on $(FS(X), d_{FS(X)})$ isometrically and properly. In addition, if Γ acts cocompactly on X , then Γ acts cocompactly on $FS(X)$.*

Now we define our flow space by

$$HFS(T_d \times \mathbb{R}) := FS(T_d) \times \mathbb{R}$$

where T_d has its natural metric with edge length 1. Since Γ has an action on both $FS(T_d)$ and \mathbb{R} , Γ will have a diagonal action on $FS(T_d) \times \mathbb{R}$ also. One can think of $HFS(T_d \times \mathbb{R})$ as the horizontal subspace of $FS(T_d \times \mathbb{R})$. In fact, there is a natural embedding of $HFS(T_d \times \mathbb{R})$ (as a topological space with product topology) into $FS(T_d \times \mathbb{R})$ defined as follows: for a generalized geodesic c on T_d , and $w \in \mathbb{R}$, we define a generalized geodesic on $T_d \times \mathbb{R}$, which maps $t \in \mathbb{R}$ to $(c(t), w) \in T_d \times \mathbb{R}$. $HFS(T_d \times \mathbb{R})$ will inherit a metric from this embedding.

Lemma 3.2. *The flow space $HFS(T_d \times \mathbb{R})$ is a proper metric space, in particular a complete metric space.*

Proof In order to prove $HFS(T_d \times \mathbb{R})$ is a proper metric space, we need to show every closed ball $B_r(c) = \{c' \mid d_{HFS(T_d \times \mathbb{R})}(c, c') \leq r\}$ in $HFS(T_d \times \mathbb{R})$ is compact. Let $\{c_i\}$ be a Cauchy sequence in the closed ball $B_r(c)$, we need to show it converges to a point in $B_r(c)$. Since the space $FS(T_d \times \mathbb{R})$ is proper, we can now assume $\{c_i\}$ converges to a point c_0 in $FS(T_d \times \mathbb{R})$. We only need to show $c_0 \in HFS(T_d \times \mathbb{R})$. Denote the projection map from $T_d \times \mathbb{R}$ to T_d as q_1 , from $T_d \times \mathbb{R}$ to \mathbb{R} as q_2 , then $c_i(t) = (q_1(c_i(t)), q_2(c_i(t)))$. Suppose $c_0 \notin HFS(T_d \times \mathbb{R})$, then $q_2(c_0(t))$ is not a constant map. Choose a big enough close interval I in \mathbb{R} such that $q_2(c_0(t))$ restricted to I is not a constant, we can assume the maximum value is A_1 , while the minimum is A_2 , where $A_1 > A_2$. Let $\delta = A_1 - A_2 > 0$ and $I_1 = \{t \in I \mid q_2(c_0(t)) > A_1 - \frac{\delta}{4}\}$, correspondingly $I_2 = \{t \in I \mid q_2(c_0(t)) < A_2 + \frac{\delta}{4}\}$. Note I_1 and I_2 are nonempty sets with measure bigger than 0. Now for any given c_i , if $q_2(c_i) \geq \frac{A_1 + A_2}{2}$, then

$$\begin{aligned}
 d_{FS(X)}(c_0, c_i) &= \int_{\mathbb{R}} \frac{d_{T_d \times \mathbb{R}}(c_0(t), c_i(t))}{2e^{|t|}} dt \\
 &\geq \int_{I_2} \frac{d_{T_d \times \mathbb{R}}(c_0(t), c_i(t))}{2e^{|t|}} dt \\
 &\geq \int_{I_2} \frac{d_{T_d \times \mathbb{R}}((q_1(c_0(t)), q_2(c_0(t))), (q_1(c_i(t)), q_2(c_i(t))))}{2e^{|t|}} dt \\
 &\geq \int_{I_2} \frac{\frac{1}{2} d_{T_d \times \mathbb{R}}(c_0(t), (q_1(c_0(t)), q_2(c_i(t))))}{2e^{|t|}} dt \quad (\text{by Corollary 2.1}) \\
 &\geq \int_{I_2} \frac{\frac{1}{2} d_{T_d \times \mathbb{R}}(c_0(t), (q_1(c_0(t)), \frac{A_1+A_2}{2}))}{2e^{|t|}} dt > 0 \quad (\text{by Lemma 2.2})
 \end{aligned}$$

The last integral is independent of c_i ; denote its value as ϵ_1 . For the same reason, if $q_2(c_i) < \frac{A_1+A_2}{2}$, there exists $\epsilon_2 > 0$ such that $d_{FS(X)}(c_0, c_i) \geq \epsilon_2$. Let $\epsilon = \min(\epsilon_1, \epsilon_2) > 0$, then $d_{FS(X)}(c_0, c_i) \geq \epsilon > 0$. Hence the sequence $\{c_i\}$ can never converge to c_0 , contradiction. \square

We define now the flow

$$\Phi : HFS(T_d \times \mathbb{R}) \times \mathbb{R} \rightarrow HFS(T_d \times \mathbb{R})$$

by $\Phi_\tau((c, w))(t) = (c(t + \tau), w)$ for $c \in FS(T_d)$ and $\tau, w, t \in \mathbb{R}$. Note Φ is a Γ -equivariant flow.

For the rest of this section, let $X = T_d \times \mathbb{R}$.

Lemma 3.3. *The flow space $HFS(X)$ has the following properties:*

- (i) Γ acts properly and cocompactly on $HFS(X)$.
- (ii) Given $C > 0$, there is only finitely many Γ orbits of periodic flow curves with period less than C (but bigger than 0).
- (iii) Let $HFS(X)^\mathbb{R}$ denote the \mathbb{R} -fixed point set, i.e., the set of points $c \in HFS(X)$ for which $\Phi_\tau(c) = c$ for all $\tau \in \mathbb{R}$, then $HFS(X) - HFS(X)^\mathbb{R}$ is locally connected.
- (iv) If we put

$$k_\Gamma := \sup\{|H| \mid H \subseteq \Gamma \text{ subgroup with finite order } |H|\};$$

$$d_{HFS(X)} := \dim(HFS(X) - HFS(X)^\mathbb{R});$$

then k_Γ and $d_{HFS(X)}$ are finite.

Proof Those properties are essentially implied by results in [4], section 1.

- (i) The Γ action on $HFS(X)$ is proper since $HFS(X)$ Γ -equivariantly embeds into $FS(X)$ and the Γ -action on $FS(X)$ is proper by Proposition 3.1. By Lemma 1.10 in [4], for any $w_0 \in \mathbb{R}$, the evaluation map $FS(T_d) \rightarrow T_d$ defined by $c \rightarrow c(w_0)$ is proper. Hence $FS(T_d) \times \mathbb{R} \rightarrow T_d \times \mathbb{R}$ is proper. This will induce a map $FS(T_d) \times_{\Gamma} \mathbb{R} \rightarrow T_d \times_{\Gamma} \mathbb{R}$. Since $T_d \times_{\Gamma} \mathbb{R}$ is compact, $FS(T_d) \times_{\Gamma} \mathbb{R}$ is compact also. One can prove this by choosing a compact fundamental domain in $T_d \times \mathbb{R}$, and using the fact that the map $FS(T_d) \times \mathbb{R} \rightarrow T_d \times \mathbb{R}$ is proper. Hence Γ acts cocompactly on $HFS(X)$.
- (ii) Note that periodic orbits in $HFS(T_d \times \mathbb{R})$ are periodic orbits in $FS(T_d \times \mathbb{R})$, which move horizontally (i.e., move along the tree direction, with \mathbb{R} coordinate fixed). Note also that the embedding of $HFS(T_d \times \mathbb{R})$ into $FS(T_d \times \mathbb{R})$ is a Γ -equivariant map, while there are only finitely many nonzero horizontal periodical geodesics on $T_d \times_{\Gamma} \mathbb{R}$ of period less than C . So there will be only finitely many nonzero horizontal periodical orbits on $HFS(T_d \times_{\Gamma} \mathbb{R})$ with period less than C . Hence the claim in (ii) now follows. See [17] for more details.
- (iii) This is because T_d is a tree, hence a CAT(0) space. In Bartels and Lück's paper [4], Proposition 2.10, they proved that for any CAT(0) space A , $FS(A) - FS(A)^{\mathbb{R}}$ is locally connected. Our flow space $HFS(X) = FS(T_d) \times \mathbb{R}$. Since the flow on our flow space only flow on the first factor, $HFS(X)^{\mathbb{R}} = FS(T_d)^{\mathbb{R}} \times \mathbb{R}$. So $HFS(X) - HFS(X)^{\mathbb{R}}$ will be locally connected as well.
- (iv) Since Γ is a torsion free group, $k_{\Gamma} = 1$. Note that any two point can be connected by a unique geodesic in the tree T_d , it is not hard to see that the flow space $FS(T_d)$ will have dimension less than 5. Therefore our flow space $HFS(X) = FS(T_d) \times \mathbb{R}$ has finite dimension, and hence $d_{HFS(X)}$ is finite. One can also consult Bartels and Lück's paper [4], they have more general results for CAT(0) spaces.

□

Remark 3.1. We define an embedding $\Psi : T_d \times \mathbb{R} \rightarrow FS(T_d) \times \mathbb{R}$, by $(z, w) \rightarrow (c_z, w)$, where c_z is the unique generalized geodesic which sends $(-\infty, 0)$ to z , and $[0, \infty)$ isometrically to the geodesic $[z, \omega)$. Recall $[z, \omega)$ is the unique geodesic connecting z and the specified infinity point ω of T_d defined in Section 2. Also, we can flow this embedding by flowing its image in $HFS(X)$; define $\Psi_{\tau}(z, w) = \Phi_{\tau}(\Psi(z, w))$. It is easy to see that Ψ_{τ} is an Γ -equivariant map.

We need the following lemma in the future.

Lemma 3.4. *Let z_0 be a fixed point in T_d , w_1, w_2 are two fixed points in \mathbb{R} , and $P_n = (z_0, \frac{w_1}{n})$, $Q_n = (z_0, \frac{w_2}{n})$, $d_X(P_1, Q_1) < D$. Then for any $\epsilon > 0$, there exists a number \tilde{N} , which depends only on ϵ , D and d , such that for any $n > \tilde{N}$*

$$d_X(P_n, Q_n) < \frac{\epsilon}{4}$$

and

$$d_{HFS(X)}(\Psi(P_n), \Psi(Q_n)) \leq \epsilon$$

Proof Choose $T = \max\{1, \ln \frac{4}{\epsilon}\}$. Since $\Psi(P_n)(T)$ and $\Psi(Q_n)(T)$ have the same T_d coordinate, by Lemma 2.3, we can choose a big enough integer N such that for any $n > N$, $d_X(\Psi(P_n)(T), \Psi(Q_n)(T)) < \frac{\epsilon}{4}$. Note \tilde{N} depends only on ϵ , D and d . Using the definition of generalized geodesic, we have $d_X(\Psi(P_n)(t), \Psi(P_n)(T)) = t - T$ and $d_X(\Psi(Q_n)(t), \Psi(Q_n)(T)) = t - T$, where $t \geq T$. Hence for any $t \geq T$, by triangle inequality, we have the following

$$\begin{aligned} & d_X(\Psi(P_n)(t), \Psi(Q_n)(t)) \\ & \leq d_X(\Psi(P_n)(t), \Psi(P_n)(T)) + d_X(\Psi(P_n)(T), \Psi(Q_n)(T)) + d_X(\Psi(Q_n)(T), \Psi(Q_n)(t)) \\ & \leq \frac{\epsilon}{4} + 2(t - T) \end{aligned}$$

On the other hand, the metric defined on $T_d \times \mathbb{R}$ is expanding in the \mathbb{R} direction when moving towards ω . Hence for any $0 \leq t \leq T$, we have

$$d_X(\Psi(P_n)(t), \Psi(Q_n)(t)) \leq d_X(\Psi(P_n)(T), \Psi(Q_n)(T)) < \frac{\epsilon}{4}$$

And for $t \leq 0$, $\Psi(P_n)(t) = \Psi(P_n)(0) = P_n$, $\Psi(Q_n)(t) = \Psi(Q_n)(0) = Q_n$, hence

$$d_X(\Psi(P_n)(t), \Psi(Q_n)(t)) = d_X(P_n, Q_n) < \frac{\epsilon}{4}.$$

Therefore, for any $n > \tilde{N}$

$$\begin{aligned} d_{HFS(X)}(\Psi(P_n), \Psi(Q_n)) &= \int_{\mathbb{R}} \frac{d_X(\Psi(P_n)(t), \Psi(Q_n)(t))}{2e^{|t|}} dt \\ &= \int_{(-\infty, 0]} \frac{d_X(\Psi(P_n)(t), \Psi(Q_n)(t))}{2e^{|t|}} dt + \int_{[0, T]} \frac{d_X(\Psi(P_n)(t), \Psi(Q_n)(t))}{2e^{|t|}} dt \\ &\quad + \int_{[T, \infty)} \frac{d_X(\Psi(P_n)(t), \Psi(Q_n)(t))}{2e^{|t|}} dt \\ &\leq \int_{(-\infty, 0]} \frac{\frac{\epsilon}{4}}{2e^{|t|}} dt + \int_{[0, T]} \frac{\frac{\epsilon}{4}}{2e^{|t|}} dt + \int_{[T, \infty)} \frac{\frac{\epsilon}{4} + 2(t - T)}{2e^{|t|}} dt \\ &= \frac{\epsilon}{4} + e^{-T} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \end{aligned}$$

Hence we proved the Lemma. \square

Because of the properties proved in Lemma 3.3, Theorem 1.4 in [5] yields a long thin cover for $HFS(X)$; i.e. the following result holds

Proposition 3.2. *There exists a natural number N , depending only on k_Γ , $d_{HFS(X)}$ and the action of Γ on an arbitrary neighborhood of $HFS(X)^\mathbb{R}$ such that for every $\lambda > 0$ there is an \mathcal{VCyc} -cover \mathcal{U} of $HFS(X)$ with the following properties:*

- (i) $\dim \mathcal{U} \leq N$;
- (ii) For every $x \in HFS(X)$ there exists $U_x \in \mathcal{U}$ such that

$$\Phi_{[-\lambda, \lambda]}(x) := \{\Phi_\tau(x) | \tau \in [-\lambda, \lambda]\} \subseteq U_x;$$

- (iii) $\Gamma \setminus \mathcal{U}$ is finite.

where \mathcal{VCyc} denote the collections of virtually cyclic subgroups of a group.

Remark 3.2. *Note that a group in the family of virtually cyclic subgroups of $\mathbb{Z}[\frac{1}{d}] \rtimes_\alpha \mathbb{Z}$ is isomorphic to either $\{0\}$ or \mathbb{Z} , which are abelian groups. So our proposition is also true with respect to the family of abelian subgroups.*

Recall that the dimension of a cover \mathcal{U} is defined to be the greatest N such that there exists $N + 1$ elements in \mathcal{U} with nonempty intersection. In general, for a collection of subgroups \mathcal{F} , we define a \mathcal{F} -cover as following.

Definition 3.3. *Let G be a group and Z be a G -space. Let \mathcal{F} be a collection of subgroups of G . An open cover \mathcal{U} of Z is called an \mathcal{F} -cover if the following three conditions are satisfied.*

- (i) For $g \in G$ and $U \in \mathcal{U}$ we have either $g(U) = U$ or $g(U) \cap U = \emptyset$;
- (ii) For $g \in G$ and $U \in \mathcal{U}$, we have $g(U) \in \mathcal{U}$;
- (iii) For $U \in \mathcal{U}$ the subgroup $G_U := \{g \in G | g(U) = U\}$ is a member of \mathcal{F} .

For a subset A of a metric space Z and $\delta > 0$, A^δ denotes the set of all points $z \in Z$ for which $d(z, A) < \delta$. Combining Lemma 3.1 and the fact that Γ acts cocompactly on $FS(T_d) \times \mathbb{R}$ (Lemma 3.3, (i)), Proposition 3.2 can be improved to the following.

Proposition 3.3. *There exists a natural number N , depending only on k_Γ , $d_{HFS(X)}$ and the action of Γ on an arbitrary neighborhood of $HFS(X)^\mathbb{R}$ such that for every $\lambda > 0$ there is a \mathcal{VCyc} -cover \mathcal{U} of $HFS(X)$ with the following properties:*

- (i) $\dim \mathcal{U} \leq N$;

- (ii) *There exists a $\delta > 0$ depends on λ such that for every $x \in HFS(X)$ there exists $U_x \in \mathcal{U}$ such that*

$$(\Phi_{[-\lambda, \lambda]}(x))^\delta \subseteq U_x;$$

- (iii) $\Gamma \setminus \mathcal{U}$ is finite.

Proof A simple modification of the argument in [5], section 1.3, page 1804-1805, yields the result. In their proof, they used a lemma (Lemma 7.2) which will be replaced by Lemma 3.1 in our case. \square

4. HYPER-ELEMENTARY SUBGROUPS OF $\mathbb{Z}_{q^s} \rtimes_\alpha \mathbb{Z}_{t_s}$

In this section, we study the hyper-elementary subgroups of $\mathbb{Z}_{q^s} \rtimes_\alpha \mathbb{Z}_{t_s}$, where d is an integer such that $|d| > 1$, q is a prime number greater than $|d| + 1$, s is a positive integer, α is multiplication by d , t_s is the order of d in the group of units of the ring \mathbb{Z}_{q^s} . We denote the group of units of a ring R by $U(R)$, hence t_s is the order of d in $U(\mathbb{Z}_{q^s})$.

Definition 4.1. A *hyper-elementary group* H is an extension of a p -group by a cyclic group of order n , where p is a prime number, $(n, p) = 1$, in other words, there exists a short exact sequence

$$1 \rightarrow C_n \rightarrow H \rightarrow G_p \rightarrow 1$$

where C_n is a cyclic group of order n , G_p is a p -group such that $(n, p) = 1$.

Note first that the order of the units group $U(\mathbb{Z}_q)$ is $q - 1$, while the order of $U(\mathbb{Z}_{q^s})$ is $q^s - q^{s-1} = q^{s-1}(q - 1)$.

Lemma 4.1. *If $d^t \equiv 1 \pmod{q}$, then $d^{tq^{s-1}} \equiv 1 \pmod{q^s}$.*

Proof We prove this by induction. For $s = 1$, this is automatically true. Now assume it is true for k , we prove it for $k + 1$. By hypothesis, $d^{tq^{k-1}} = mq^k + 1$, where m is an integer. So $d^{tq^k} = (mq^k + 1)^q$. Expanding the right side, it is easy to see that $d^{tq^k} \equiv 1 \pmod{q^{k+1}}$. \square

If we assume the order of d in $U(\mathbb{Z}_q)$ is t_1 , and the order of d in $U(\mathbb{Z}_{q^s})$ is t_s , then $t_1 \mid q - 1$, and $t_s \mid q^{s-1}(q - 1)$. Let $t_s = m_s q^{k_s}$, where $(m_s, q) = 1$. Then what the previous lemma says is that $m_s \mid t_1$ and $k_s \leq s - 1$. In the following lemma, we prove $m_s = t_1$.

Lemma 4.2. *Assume t_1 is the order of d in $U(\mathbb{Z}_q)$, $t_s = m_s q^{k_s}$ is the order of d in $U(\mathbb{Z}_{q^s})$, where $(m_s, q) = 1$, then $m_s = t_1$.*

Proof We prove it by contradiction. Assume $m_s \neq t_1$, denote $d^{m_s} = a$, then a is not equal to 1 modulo q , since $m_s | t_1$. But since the order of d in $U(\mathbb{Z}_{q^s})$ is $m_s q^{k_s}$, we have $d^{m_s q^{k_s}} \equiv a^{q^{k_s}} \equiv 1 \pmod{q^s}$. This means $a^{q^{k_s}} \equiv 1 \pmod{q}$, which is not true. In fact, by Fermat's theorem, $a^{q^{k_s}} \equiv a^{q^{k_s-1}} \equiv \dots a \pmod{q}$. \square

We list some basic formulas we are going to use frequently:

- (1) $(a, b)(a', b') = (a + d^{-b}a', b + b')$;
- (2) $(a, b)^n = ((1 + d^{-b} + \dots + d^{-(n-1)b})a, nb)$;
- (3) $(a, b)^{-1} = (-d^b a, -b)$;
- (4) $(x, y)(a, b)(x, y)^{-1} = ((1 - d^{-b})x + d^{-y}a, b)$;

where $(a, b), (a', b'), (x, y) \in \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$, n is a positive integer and d^{-b} is the inverse element of d^b in the unit group $U(\mathbb{Z}_{q^s})$, etc.

Lemma 4.3. *For any $(a, b) \in \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$, if $t_1 \nmid b$, then (a, b) can be conjugate to $(0, b)$.*

Proof Note first if $t_1 \nmid b$, then $d^{-b} \not\equiv 1 \pmod{q}$, hence $1 - d^{-b}$ is a unit in \mathbb{Z}_{q^s} . Since $(x, y)(a, b)(x, y)^{-1} = ((1 - d^{-b})x + d^{-y}a, b)$, if $1 - d^{-b}$ is a unit in \mathbb{Z}_{q^s} , then we can find x, y , such that $(1 - d^{-b})x + d^{-y}a = 0$, which means we can conjugate (a, b) to $(0, b)$. In fact, we can always take $y = 0$. \square

Theorem 4.1. *For $s > q$ and q a prime number bigger than $|d| + 1$, every hyper-elementary subgroup H of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ can be conjugated to a subgroup of one of the following three types of subgroups:*

- type 1: $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{q^{k_s}}$;
- type 2: $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_1}$;
- type 3: $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$.

where α is multiplication by d , t_1 is the order of d in the group of units $U(\mathbb{Z}_q)$ and t_s is the order of d in the group of units $U(\mathbb{Z}_{q^s})$, $t_s = t_1 q^{k_s}$.

Remark 4.1. *The homomorphism $\alpha : \mathbb{Z}_{q^{k_s}} \rightarrow \text{Aut}(\mathbb{Z}_{q^s})$ in $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{q^{k_s}}$ is the restriction of the homomorphism $\alpha : \mathbb{Z}_{t_s} \rightarrow \text{Aut}(\mathbb{Z}_{q^s})$, to the subgroup $\mathbb{Z}_{q^{k_s}} \subseteq \mathbb{Z}_{t_s}$. There is a similar remark for $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_1}$.*

Proof Denote the cyclic part of the hyper-elementary subgroup H by C . If $s > q$, then $q^s > |d|^{q-1}$, which implies $k_s \geq 1$.

First, if C is a trivial group, then H is just a p -group. By Sylow's theorem, any p -group can be conjugate to a subgroup of a maximal p -group. Note that the order of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ is $q^s \cdot t_s = q^s \cdot t_1 q^{k_s} = t_1 q^{s+k_s}$, where t_1 and q are coprime, hence any p -group in $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ can be conjugated to a subgroup of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{q^{k_s}}$ (type 1), or $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_1}$ which is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ (type 3).

Now assume C is non-trivial, say C is generated by $(a, b) \in \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$.

If $b = 0$, C will have order a power of q , then since the order of H 's p -group has to be coprime to q , the p -group's order has to divide t_1 , which means H is a subgroup of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_1}$ (type 2). So for the rest of the proof, we will assume that $b \neq 0$.

If $a = 0$, C lies in $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$. If the p -group part of H is trivial, then we are in type 3. Otherwise in order for C to be a normal subgroup, for any $(x, y) \in H$, $(x, y)(0, b)(x, y)^{-1} = ((1 - d^{-b})x, b)$ has to lie in C as well, which means $(1 - d^{-b})x$ has to be equal to zero in \mathbb{Z}_{q^s} . Note first that this means C lies in the center of H . If $x = 0$ always, then H is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ (type 3). If $x \neq 0$ for some (x, y) which lies in the p -group of H , then at least, $(1 - d^{-b}) \equiv 0 \pmod{q}$ and $x \equiv 0 \pmod{q}$. Hence $d^{-b} \equiv 1 \pmod{q}$, therefore $t_1 \mid b$. So the order of the cyclic group is a power of q , while the order of H 's p -group has to be coprime to q . By Sylow's theorem, H 's p -group can be conjugate to a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_1}$, hence it is actually a cyclic subgroup. This implies the hyper-elementary subgroup H is an abelian group since C is in the center of H . Combined with the fact that the order of the cyclic group and p -group are coprime, we have H is again a cyclic group, say generated by (a', b') , where b' is not zero. The case $a' = 0$ is contained in type 3. For the case $a' \neq 0$, if $t_1 \nmid b'$, by Lemma 4.3, (a', b') can be conjugate to $(0, b')$ (type 3). When $t_1 \mid b'$, H will lie in $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{q^{k_s}}$ (type 1).

Now suppose both a and b are not zero. If t_1 does not divide b , then by Lemma 4.3, we can conjugate (a, b) to $(0, b)$, which can be included in the $a = 0$ case. When $t_1 \mid b$, then $d^{-b} \equiv 1 \pmod{q}$. And b lies in $\mathbb{Z}_{q^{k_s}} (\subseteq \mathbb{Z}_{t_s})$, hence the order of C is a power of q since (a, b) generates C in $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$. Let b 's order in \mathbb{Z}_{t_s} be q^r . Since H 's p -group has order coprime to q , again by Sylow's theorem, it can be conjugate to a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_1}$. Therefore we can assume the generator of H 's p -group to be $(0, y)$. If $y \neq 0$, then t_1 does not divide y , so d^y is not equal to 1 modulo q . On the other hand, since $(0, y)(a, b)(0, y)^{-1} = (d^{-y}a, b)$, $(d^{-y}a, b)$ has to lie in C , which means $(d^{-y}a, b) = (a, b)^n$ for some n . Recall from formula (2) that $(a, b)^n = ((1 + d^{-b} + \dots + d^{-(n-1)b})a, nb)$. In order for $nb = b$, which is the same as $(n-1)b = 0$, b 's order q^r has to divide $n-1$. For such n , $\Sigma = 1 + d^{-b} + \dots + d^{-(n-1)b}$

will equal to 1 modulo q ; note $d^{-b} \equiv 1 \pmod{q}$. Consequently, $d^{-y} - \Sigma$ is a unit in $\mathbb{Z}_{q^{k_s}}$ and $(d^{-y} - \Sigma)a = 0 \in \mathbb{Z}_{q^{k_s}}$, which is a contradiction since $a \neq 0$. Therefore, y has to be 0 and H 's p -group has to be trivial, hence H will now be a subgroup of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{q^{k_s}}$ (type 1). \square

Corollary 4.1. *For any $n > 1$, q a prime number greater than $|d|^n$ and $s > q$, each hyper-elementary subgroup of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ has index greater than n . Recall α is multiplication by d and t_s is the order of d in the group of units $U(\mathbb{Z}_{q^s})$.*

Corollary 4.2. *For any $n > 1$, q a prime number greater than $|d|^n$ and $s > q$, each hyper-elementary subgroup of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ is conjugate to a subgroup H , such that one of the following is true:*

- (1) *the index $[\pi(H), \mathbb{Z}_{t_s}] \geq n$, where $\pi : \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s} \rightarrow \mathbb{Z}_{t_s}$ is the natural epimorphism.*
- (2) *H is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$, and $q^s \geq n$.*

Proof If $q > |d|^n$, then the order of d in \mathbb{Z}_q is greater than n . Also $s > q$, so $k_s \geq 1$. Hence the order of d in \mathbb{Z}_{q^s} is $t_s = t_1 q^{k_s}$, where $t_1 > n$, $q > n$, and $k_s \geq 1$. Now by theorem 4.1 the two corollaries follow easily. \square

5. PROOF OF THE MAIN THEOREM

In this section we prove our main theorem. Our strategy is to prove that $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$ is in fact a Farrell-Hsiang group, defined by Bartels and Lück in [3]. Recall that d is a positive integer greater than 1 and α is multiplication by d .

Definition 5.1. *Let \mathcal{F} be a family of subgroups of the finitely generated group G . We call G a **Farrell-Hsiang Group** with respect to the family \mathcal{F} if the following holds for a fixed word metric d_G :*

There exists a fixed natural number N such that for every natural number n there is a surjective homomorphism $\alpha_n : G \rightarrow F_n$ with F_n a finite group such that the following condition is satisfied. For every hyper-elementary subgroup H of F_n we set $\bar{H} := \alpha_n^{-1}(H)$ and require that there exists a simplicial complex E_H of dimension at most N with a cell preserving simplicial \bar{H} -action whose stabilizers belong to \mathcal{F} , and an \bar{H} -equivariant map $f_H : G \rightarrow E_H$ such that $d_G(g_0, g_1) < n$ implies $d_{E_H}^1(f_H(g_0), f_H(g_1)) < \frac{1}{n}$ for all $g_0, g_1 \in G$, where $d_{E_H}^1$ is the l^1 -metric on E_H .

Remark 5.1. As A. Bartels, T. Farrell and W. Lück pointed out in [2], Remark 1.15, in order to check a group G is a Farrell-Hsiang group, we only need to check these conditions for one hyper-elementary subgroup in every conjugacy class of hyper-elementary subgroups of F_n .

With this definition, they proved the following theorem:

Theorem 5.1. *Let G be a Farrell-Hsiang group with respect to the family \mathcal{F} . Then G satisfies the K-theoretic and L-theoretic Farrell-Jones Conjecture with respect to the family \mathcal{F} .*

Before we go to our proof, we need to recall the transitivity principle. (see for example [2], Theorem 1.11.)

Lemma 5.1 (Transitivity principle). *Let $\mathcal{F} \subseteq \mathcal{H}$ be two families of subgroups of G . Assume that G satisfies the K-Theoretic Farrell-Jones conjecture with respect to \mathcal{H} and that each $H \in \mathcal{H}$ satisfies the K-theoretic Farrell-Jones conjecture with respect to \mathcal{F} . Then G satisfies the K-theoretic Farrell-Jones conjecture with respect to \mathcal{F} . The same is true for the L-theoretic Farrell-Jones conjecture.*

Since each virtually cyclic subgroup of Γ is either isomorphic to $\{0\}$ or \mathbb{Z} , the family of all virtually cyclic subgroups of Γ is contained in the family of all abelian subgroups of Γ . And since both the K-theoretic and L-theoretic Farrell-Jones conjecture have been verified for abelian groups with respect to the family of virtually cyclic subgroups, by the transitivity principle, in order to prove our main theorem, we only need to prove it with respect to the family of abelian subgroups.

We begin to prove the main theorem now. Choose $N > 0$ to be the number which appears in Proposition 3.3, note N is independent of λ . First, there is a quotient map $\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z} \rightarrow \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$, where t_s is the order of d in the unit group of \mathbb{Z}_{q^s} . Let F_n be $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$, where we choose q to be a prime number bigger than d^n and $s > q$ (Hence Corollary 4.2 holds; we will manipulate s more in the future to get more control.) Let α_n be the quotient map. Also let H be a hyper-elementary subgroup of F_n and $\bar{H} = \alpha_n^{-1}(H)$. The metric on Γ is inherited from the orbit embedding $\eta : \Gamma \rightarrow T_d \times \mathbb{R}$ where $\eta(g) = gx_0$, $g \in \Gamma$ and $x_0 = (P_0, 0) \in T_d \times \mathbb{R}$ is the base point.

By Corollary 4.2, For any $n > 0$, q a prime number greater than d^{4n^2} and $s > q$, each hyper-elementary subgroup of $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ is conjugate to a subgroup H , such that one of the following is true:

Case (1), $[\pi(H), \mathbb{Z}_{t_s}] \geq 4n^2$, where $\pi : \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s} \mapsto \mathbb{Z}_{t_s}$ is the natural epimorphism.
Case (2), H is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$

We proceed under the assumption that case (1) holds; i.e. we assume $[\pi(H), \mathbb{Z}_{t_s}] = m \geq 4n^2$. Choose E_H to be the real line \mathbb{R} (dimension 1), with the simplicial structure with $m\mathbb{Z}$ as vertices. There is a Γ action on \mathbb{R} by $(x, k)y = y + k$ for $(x, k) \in$

$\mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$, $y \in \mathbb{R}$. Note Γ does not act simplicially on \mathbb{R} , but \bar{H} does. The stabilizers of a vertex or an edge is $\bar{H} \cap \mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \{0\}$, hence an abelian group. Define $f_H : \Gamma \rightarrow E_H$, by mapping $(x, k) \in \mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}$ to $k \in \mathbb{R}$, f_H is an \bar{H} equivariant map. We need to prove for $g, h \in \Gamma$, if $d_G(g, h) < n$, then $d_{E_H}^1(f_H(g), f_H(h)) < \frac{1}{n}$. Let $g = (x_1, k_1)$ and $h = (x_2, k_2)$, then $f_H(g) = k_1$ and $f_H(h) = k_2$. Hence $d(f_H(g), f_H(h)) = |k_1 - k_2|$. On the other hand, $d_G(g, h) = d(g(P_0, 0), h(P_0, 0)) = d(h^{-1}g(P_0, 0), (P_0, 0))$. Note $h^{-1}g = (d^{-k_2}(x_1 - x_2), k_1 - k_2)$, while $h^{-1}g$ acts on $(P_0, 0)$ as the matrix $\begin{pmatrix} d^{-(k_1-k_2)} & d^{-k_2}(x_1 - x_2) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d^{-k_2}(x_1 - x_2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-(k_1-k_2)} & 0 \\ 0 & 1 \end{pmatrix}$. By checking the action on P_0 , one sees that $h^{-1}g$ moves P_0 at least distance $|k_1 - k_2|$ away. By Lemma 2.1, $|k_1 - k_2| < d(h^{-1}g(P_0, 0), (P_0, 0)) = d(g, h)$. So $d(f_H(g), f_H(h)) < d(g, h) < n$. One can easily check now that $d_{E_H}^1(f_H(g), f_H(h)) < \frac{1}{n}$ since the simplicial structure we put on \mathbb{R} has edge length greater than m while $m > 4n^2$. Hence we have completed the proof in this case.

Now we proceed to Case(2). In this case H is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$. Denote $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$ by K , then $\bar{K} = \pi^{-1}(K) = (q^s \mathbb{Z}[\frac{1}{d}]) \rtimes_{\alpha} \mathbb{Z}$, $\bar{H} \subseteq \bar{K}$. Define a monomorphism $\varphi : \Gamma \rightarrow \Gamma$ by

$$\varphi((x, k)) = (q^s x, k), \text{ for } (x, k) \in \mathbb{Z}[\frac{1}{d}] \rtimes_{\alpha} \mathbb{Z}.$$

Note that its image is \bar{K} ; hence $\varphi^{-1} : \bar{K} \rightarrow \Gamma$ is well defined. Define a self homeomorphism $F_{q^s} : T_d \times \mathbb{R} \rightarrow T_d \times \mathbb{R}$ by

$$F_{q^s}(z, w) = \begin{pmatrix} q^s & 0 \\ 0 & 1 \end{pmatrix} (z, w), \text{ for } (z, w) \in T_d \times \mathbb{R}$$

where the action of $\begin{pmatrix} q^s & 0 \\ 0 & 1 \end{pmatrix}$ on $(z, w) \in T_d \times \mathbb{R}$ is the diagonal action explained

in section 2. In fact, $\begin{pmatrix} q^s & 0 \\ 0 & 1 \end{pmatrix}$ acts on the tree T_d as an isometry which fixes

the line L_0 while it acts on \mathbb{R} by $\begin{pmatrix} q^s & 0 \\ 0 & 1 \end{pmatrix} w = q^s w$ for $w \in \mathbb{R}$. It is easy to see that F_{q^s} is φ (semi)-equivariant since $(x, k) \in \Gamma$ acts on $T_d \times \mathbb{R}$ as the matrix $\begin{pmatrix} d^{-k} & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} q^s & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-k} & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d^{-k} & q^s x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^s & 0 \\ 0 & 1 \end{pmatrix}$. Recall F_{q^s} is φ (semi)-equivariant if $F_{q^s}(g(z, w)) = \varphi(g)F_{q^s}((z, w))$ for every $g \in \Gamma$.

Now choose λ to be n , then by Proposition 3.3, we have a \mathcal{VCyc} -cover \mathcal{U} of dimension N of $HFS(T_d \times \mathbb{R})$ satisfying the following. There exists $\delta > 0$ which depends on λ such that for any point $x \in HFS(T_d \times \mathbb{R})$, there exists $U_x \in \mathcal{U}$ such

that $(\Phi_{[-n,n]}(x))^\delta \subseteq U_x$. We need to define a \bar{H} -equivariant map from $T_d \times \mathbb{R}$ to $HFS(T_d \times \mathbb{R})$. Using this map we can pull the \mathcal{VCyc} -cover \mathcal{U} on $HFS(T_d \times \mathbb{R})$ back to $T_d \times \mathbb{R}$ and get a \mathcal{VCyc} -cover on $T_d \times \mathbb{R}$. Consider the following diagram

$$\Gamma \xrightarrow{\eta} T_d \times \mathbb{R} \xrightarrow{F_{q^s}^{-1}} T_d \times \mathbb{R} \xrightarrow{\Psi_\tau} HFS(T_d \times \mathbb{R})$$

where η is the inclusion map given by the orbit of the base point, and $\Psi_\tau = \Phi_\tau \circ \Psi$ as defined in remark 3.1. In order to guarantee the composition is \bar{H} equivariant, we change the action of \bar{H} on the second $T_d \times \mathbb{R}$ (i.e., the image of $F_{q^s}^{-1}$) by defining $\bar{h} \bullet (z, w) = \varphi^{-1}(\bar{h})(z, w)$, for any $\bar{h} \in \bar{H}$, $(z, w) \in T_d \times \mathbb{R}$. We also define an \bar{H} action on $HFS(T_d \times \mathbb{R})$ using the same method, by composing the action of Γ with φ^{-1} . Now using the composite $\Psi_\tau \circ F_{q^s}^{-1}$, we pull back the cover on $HFS(T_d \times \mathbb{R})$ to $T_d \times \mathbb{R}$ and denote the nerve of this cover by $E(\lambda)$. It is a simplicial \bar{H} -complex of dimension N whose stabilizers belong to \mathcal{VCyc} . Also there is a canonical map from $T_d \times \mathbb{R}$ to $E(\lambda)$ (see for example, [6], section 4.1), denote this map as \hat{f}_H . Now for suitable choices of τ and q^s , $f_H = \hat{f}_H \circ \eta$ will be the map used to make Γ a Farrell-Hsiang group with respect to the family of abelian subgroups. In fact if we choose $\tau = \ln n - \ln \delta + n$, and rechoose s so that $s > q$ and $q^s > \bar{N}$, where \bar{N} is determined by Lemma 3.4 choosing $\epsilon = \frac{\delta^2}{2ne^n}$, $D = 2n$. Note that d is fixed once our group Γ is fixed.

Now we begin to prove that with these choices f_H indeed works. Let $g_0, g_1 \in \Gamma \subseteq T \times \mathbb{R}$ with $d(g_0, g_1) < n$, where the inclusion is by the embedding η . And let $g_2 = (q_1(g_0), q_2(g_1))$, recall that q_1, q_2 are projections from $T_d \times \mathbb{R}$ to T_d or \mathbb{R} respectively. By Lemma 2.1 and Corollary 2.1, we have $d(g_2, g_1) < d(g_0, g_1) < n$, $d(g_2, g_0) < 2d(g_0, g_1) < 2n$. Applying $F_{q^s}^{-1}$, by Lemma 3.4 and our choice of ϵ and D , $d(\Psi_0(F_{q^s}^{-1}(g_0)), \Psi_0(F_{q^s}^{-1}(g_2))) < \epsilon = \frac{\delta^2}{2ne^n}$. On the other hand, $d(g_2, g_1) = d(F_{q^s}^{-1}(g_2), F_{q^s}^{-1}(g_1))$, which is the distance between $q_1(g_2)$ and $q_1(g_0)$ in the tree T_d by Lemma 2.1, since g_2 and g_1 has the same \mathbb{R} coordinate. For convenience we will denote $F_{q^s}^{-1}(g_i)$ by \bar{g}_i , $i = 0, 1, 2$, then $d(\bar{g}_1, \bar{g}_2) < n$ and $d(\Psi_0(\bar{g}_0), \Psi_0(\bar{g}_2)) < \epsilon$.

As shown in Figure 2 there are two subcases to consider depending on the positions of \bar{g}_1 and \bar{g}_2 . Since $\tau = \ln n - \ln \delta + n$, by Lemma 3.1,

$$d(\Psi_\tau(\bar{g}_0), \Psi_\tau(\bar{g}_2)) \leq e^{|\tau|} d(\Psi_0(\bar{g}_0), \Psi_0(\bar{g}_2)) < \frac{ne^n}{\delta} \cdot \frac{\delta^2}{2ne^n} = \frac{\delta}{2}$$

In **subcase 1**, we first assume \bar{g}_1 is ahead of \bar{g}_2 as in Figure 2. Let $\hat{d} = d(\bar{g}_1, \bar{g}_2)$, then there exists an element of the cover \mathcal{U} , say U , such that $(\Psi_{[\tau-n, \tau+n]}(\bar{g}_2))^\delta \subseteq U$ by proposition 3.3; hence $\Psi_{\tau+\hat{d}}(\bar{g}_2) \in U$ since $\hat{d} < n$. On the other hand, as

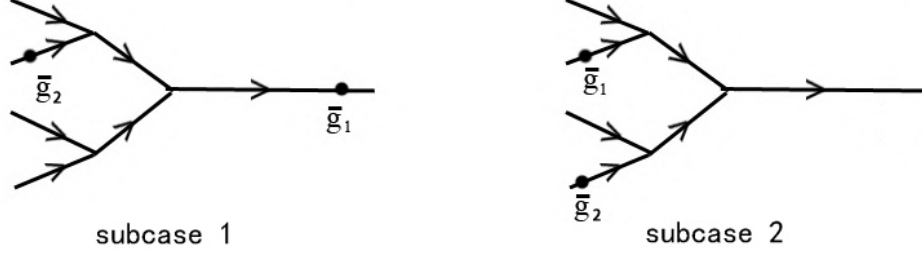


FIGURE 2. Two subcases

generalized geodesics, $\Psi_{\tau+\hat{d}}(\bar{g}_2)$ and $\Psi_\tau(\bar{g}_1)$ coincide for $t > -\tau$. In fact $\Psi_{\tau+\hat{d}}(\bar{g}_2)(t)$ maps $(-\infty, -\tau - \hat{d})$ to the point \bar{g}_2 and $[-\tau - \hat{d}, \infty)$ isometrically to the geodesic starting with the point \bar{g}_2 and ending with the infinity point ω ; while $\Psi_\tau(\bar{g}_1)(t)$ maps $(-\infty, -\tau]$ to \bar{g}_1 and $[-\tau, \infty)$ isometrically to the geodesic starting with the point \bar{g}_1 and ending with the infinity point ω . Hence

$$\begin{aligned}
d_{FS(X)}(\Psi_{\tau+\hat{d}}(\bar{g}_2), \Psi_\tau(\bar{g}_1)) &= \int_{\mathbb{R}} \frac{d(\Psi_{\tau+\hat{d}}(\bar{g}_2), \Psi_\tau(\bar{g}_1))}{2e^{|t|}} dt \\
&= \int_{(-\infty, -\tau-\hat{d}]} \frac{d(\Psi_{\tau+\hat{d}}(\bar{g}_2), \Psi_\tau(\bar{g}_1))}{2e^{|t|}} dt + \int_{[-\tau-\hat{d}, -\tau]} \frac{d(\Psi_{\tau+\hat{d}}(\bar{g}_2), \Psi_\tau(\bar{g}_1))}{2e^{|t|}} dt \\
&\quad + \int_{[-\tau, \infty]} \frac{d(\Psi_{\tau+\hat{d}}(\bar{g}_2), \Psi_\tau(\bar{g}_1))}{2e^{|t|}} dt \\
&< \int_{(-\infty, -\tau-\hat{d}]} \frac{d(\bar{g}_2, \bar{g}_1)}{2e^{-t}} dt + \int_{[-\tau-\hat{d}, -\tau]} \frac{d(\bar{g}_2, \bar{g}_1)}{2e^{-t}} dt + 0 \\
&= \int_{(-\infty, -\tau-\hat{d}]} \frac{\hat{d}}{2e^{-t}} dt + \int_{[-\tau-\hat{d}, -\tau]} \frac{\hat{d}}{2e^{-t}} dt \\
&= \frac{1}{2} \hat{d} e^{-\tau} \leq \frac{n}{2e^\tau} = \frac{n}{\frac{2e^n(n)}{\delta}} = \frac{\delta}{2e^n} < \frac{\delta}{2}.
\end{aligned}$$

Therefore, we have proven that $\Psi_\tau(F_{q^s}^{-1}(g_0)), \Psi_\tau(F_{q^s}^{-1}(g_1)) \in (\Psi_{[\tau-n, \tau+n]}(\bar{g}_2))^\delta \subseteq U$. If \bar{g}_2 is ahead of \bar{g}_1 , then the same calculation shows that

$$d_{FS(X)}(\Psi_{\tau+\hat{d}}(\bar{g}_1), \Psi_\tau(\bar{g}_2)) < \frac{\delta}{2}$$

Consequently

$$d_{FS(X)}(\Psi_{\tau+\hat{d}}(\bar{g}_1), \Psi_\tau(\bar{g}_0)) \leq d_{FS(X)}(\Psi_{\tau+\hat{d}}(\bar{g}_1), \Psi_\tau(\bar{g}_2)) + d_{FS(X)}(\Psi_\tau(\bar{g}_0), \Psi_\tau(\bar{g}_2))$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

Therefore, $\Psi_\tau(F_{q^s}^{-1}(g_0)), \Psi_\tau(F_{q^s}^{-1}(g_1)) \in (\Psi_{[\tau-n, \tau+n]}(\bar{g}_1))^\delta$ which will be contained in some member of the cover \mathcal{U} by proposition 3.3. Hence if we pull back the cover \mathcal{U} via $\Psi_\tau \circ F_{q^s}^{-1}$, we get a cover \mathcal{V} of $T_d \times \mathbb{R}$ and g_0 and g_1 will lie in the same member of \mathcal{V} .

For **subcase 2**, note that $\Psi(\bar{g}_1)$ and $\Psi(\bar{g}_2)$ as generalized geodesics in $T_d \times \mathbb{R}$ will meet; suppose $h \in T_d \times \mathbb{R}$ is the first point where they meet. Let $d_i = d(\bar{g}_i, h)$ for $i = 1, 2$. Note $d_1 + d_2 < n$. We can assume $d_1 \geq d_2$ and let $\hat{d} = d_1 - d_2$, $\hat{d} < n$. Now we are almost in the same situation as **subcase 1**. The generalized geodesics, $\Psi_\tau(\bar{g}_1)$ and $\Psi_{\tau+\hat{d}}(\bar{g}_2)$ will coincide for $t > -\tau + d_1$. And the same calculation shows that $d_{FS(X)}(\Psi_\tau(\bar{g}_1), \Psi_{\tau+\hat{d}}(\bar{g}_2)) < \frac{n}{2e^{\tau-d_1}} < \frac{n}{2\frac{n}{\delta}e^{-d_1}} < \frac{\delta}{2}$. Therefore, $d_{FS(X)}(\Psi_{\tau+\hat{d}}(\bar{g}_2), \Psi_\tau(F_{q^s}^{-1}(g_1))) < \delta$, hence there exists an element U of the cover \mathcal{U} such that $\Psi_\tau(F_{q^s}^{-1}(g_0)), \Psi_\tau(F_{q^s}^{-1}(g_1)) \in (\Psi_{[\tau-n, \tau+n]}(\bar{g}_0))^\delta \subseteq U$. Consequently if we pull back the cover \mathcal{U} , g_0 and g_1 will lie in the same member of this new cover \mathcal{V} of $T_d \times \mathbb{R}$.

So far, we have proved that there exists a \mathcal{VCyc} -cover \mathcal{V} of $T_d \times \mathbb{R}$, such that given any $g_0, g_1 \in \Gamma$ with $d(g_0, g_1) < n$, there exists an member of this cover containing both g_0 and g_1 . In fact, our proof also works for any two points in $T_d \times \mathbb{R}$ with distance less than n . Furthermore, our proof shows that any ball with radius less than $\frac{n}{2}$ will lie in some element of the cover. Hence we can apply the following lemma from [6], proposition 5.3, page 47.

Lemma 5.2. *Let $X = (X, d)$ be a metric space and $\beta \geq 1$. Suppose \mathcal{U} is an open cover of X of dimension less than or equal to N with the following property:*

For every $x \in X$, there exists $U \in \mathcal{U}$ such that the β ball around x lies in U .

Then if we denote the nerve of the cover as $N(\mathcal{U})$, the canonical map $\rho : X \rightarrow N(\mathcal{U})$ has the following contracting property. If $d(x, y) \leq \frac{\beta}{4N}$, then

$$d^1(\rho(x), \rho(y)) \leq \frac{16N^2}{\beta} d(x, y).$$

By Proposition 3.3, the dimension of the cover is less than a fixed number N in our situation. For any $g_0, g_1 \in \Gamma$ such that $d(g_0, g_1) < n$, choose $\beta = 16N^2n^2$, i.e., in the arguments of Case(2) from page 18 to page 21 before Lemma 5.2, replace n by $\bar{n} = 32N^2n^2$, hence any ball of radius $16N^2n^2$ will lie in the same element of the cover. Then $d(g_0, g_1) < n < 4Nn^2 = \frac{\beta}{4N}$, by Lemma 5.2, $d^1(f_H(g_0), f_H(g_1)) \leq \frac{16N^2}{\beta} d(g_0, g_1) = \frac{16N^2}{16N^2n^2} d(g_0, g_1) \leq \frac{1}{n^2} n = \frac{1}{n}$. Hence we have finished our proof.

6. FURTHER RESULTS

In this section we extend our results on the Farrell-Jones conjecture to more general groups. As in section 4, we denote the unit group of a ring R by $U(R)$.

Let $\hat{\Gamma}$ be the following matrix subgroup of $GL_2(\mathbb{Z}[\frac{1}{p}])$, where p is a prime number

$$\hat{\Gamma} = \left\{ \begin{pmatrix} \pm p^{-k} & x \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \text{ and } x \in \mathbb{Z}[\frac{1}{p}] \right\}$$

Note that the Baumslag-Solitar group $\Gamma = \mathbb{Z}[\frac{1}{p}] \rtimes \mathbb{Z}$ we studied before, is a normal subgroup of index 2 in $\hat{\Gamma}$ and that $\hat{\Gamma} = \text{Aff}(\mathbb{Z}[\frac{1}{p}])$; i.e. $\hat{\Gamma}$ is the full affine group of the ring $\mathbb{Z}[\frac{1}{p}]$. More precisely, $\hat{\Gamma}$ is isomorphic $\mathbb{Z}[\frac{1}{p}] \rtimes U(\mathbb{Z}[\frac{1}{p}])$ and $U(\mathbb{Z}[\frac{1}{p}]) = \{\pm p^{-k} \mid k \in \mathbb{Z}\} \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Note that virtually cyclic groups are virtually abelian while the Farrell-Jones conjecture is true for virtually abelian groups. Hence by the transitivity principle (Lemma 5.1), if we can prove $\hat{\Gamma}$ is a Farrell-Hsiang group with respect to the family of virtually abelian subgroups, then the Farrell-Jones conjecture is true for $\hat{\Gamma}$. By checking our proof for Γ , one sees that the only thing we need to take care of is section 4.

For q a prime number much bigger than p , define U_{q^s} to be the subgroup of $U(\mathbb{Z}_{q^s})$ generated by -1 and p . Note since q is an odd prime number, $U(\mathbb{Z}_{q^s})$ is a cyclic group (see for example [15], chapter 4), hence U_{q^s} is also a cyclic group, denote its generator by $d \in U(\mathbb{Z}_{q^s})$ and its order by t'_s . Then $t'_s = t_s$ or $2t_s$, where $t_s = t_1 q^{ks}$ is the order of p in $U(\mathbb{Z}_{q^s})$ as defined in section 4. Hence, t'_1 , the order of d in \mathbb{Z}_q , equals to either t_1 or $2t_1$, and $t'_s = t'_1 q^{ks}$. Note that U_{q^s} as a subgroup of $U(\mathbb{Z}_{q^s})$ will have a canonical action on \mathbb{Z}_{q^s} via multiplication, hence the semidirect product $\mathbb{Z}_{q^s} \rtimes U_{q^s}$ is well defined. And there is a quotient homomorphism $\alpha_n : \mathbb{Z}[\frac{1}{p}] \rtimes U(\mathbb{Z}[\frac{1}{p}]) \rightarrow \mathbb{Z}_{q^s} \rtimes U_{q^s}$.

Lemma 6.1. *For any given integer $n > 0$, let q be a prime greater than p^n and $s > q$, then every hyper-elementary subgroup H of $\mathbb{Z}_{q^s} \rtimes U_{q^s} = \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t'_s}$ is conjugate to a subgroup \bar{H} , such that one of the following is true:*

- (1) *the index $[\pi'(\bar{H}), \mathbb{Z}_{t'_s}] \geq n$, where $\pi' : \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t'_s} \mapsto \mathbb{Z}_{t'_s}$ is the natural epimorphism.*
- (2) *\bar{H} is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t'_s}$, and $q^s \geq n$.*

Remark 6.1. *If we write $\mathbb{Z}_{q^s} \rtimes U_{q^s}$ as $\mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t'_s}$, then α is a multiplication by d , where $d \in U(\mathbb{Z}_{q^s})$ generates U_{q^s} .*

Proof \mathbb{Z}_{t_1} is a subgroup of index 1 or 2 in $\mathbb{Z}_{t'_1}$ depending on whether $t'_1 = t_1$ or $2t_1$. Likewise \mathbb{Z}_{t_s} is a subgroup of index 1 or 2 in $\mathbb{Z}_{t'_s}$ depending on whether $t'_s = t_s$ or $2t_s$.

Let $H' = H \cap \mathbb{Z}_{q^s} \rtimes \mathbb{Z}_{t_s}$, then H' is a subgroup of index 1 or 2 of H . If the index is 1, then we are done by Corollary 4.2. Hence we can assume the index is 2; note that H' is also a hyper-elementary group in this case. Therefore by Corollary 4.2, either

(1') $[\pi'(H'), \mathbb{Z}_{t_s}] \geq n$ or

(2') H' can be conjugate to a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s} \subseteq \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s} \subseteq \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t'_s}$.

It is clear that (1') \implies (1). Therefore we assume (2') occurs. And hence we can also (after this conjugation) assume that H' is a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t_s}$. Therefore the $\text{Ker}(\pi'|_H)$ is either $\{0\}$ or \mathbb{Z}_2 . But the order of $\text{Ker}(\pi'|_H)$ has to divide q , hence $\text{Ker}(\pi'|_H) = \mathbb{Z}_2$ is impossible. Hence $\pi' : H \rightarrow \hat{H} := \pi'(H) \subseteq \mathbb{Z}_{t'_s}$ is an isomorphism. So H is a cyclic group with generator say $(a, b) \in \mathbb{Z}_{q^s} \rtimes_{\alpha} \mathbb{Z}_{t_s}$. Now note Lemma 4.3 implies if $t'_1 \nmid b$, then H can be conjugate to a subgroup of $\{0\} \rtimes_{\alpha} \mathbb{Z}_{t'_s}$. If $t'_1 \mid b$, then order of b divides q^r . But the order of b is the same as $|H|$, and $2 \mid |H|$, which is a contradiction. \square

With this Lemma, one can now check that our methods for proving the Farrell-Jones conjecture for Γ can be extended to $\hat{\Gamma}$.

Proposition 6.1. *The K- and L-theoretic Farrell-Jones conjecture is true for $\hat{\Gamma} = \text{Aff}(\mathbb{Z}[\frac{1}{p}])$.*

Corollary 6.1. *The K- and L-theoretic Farrell-Jones conjecture is true for the Baumslag-Solitar group $BS(1, d)$, where $d < -1$.*

Proof Note for $d < -1$, $BS(1, d)$ is a subgroup of $\hat{\Gamma}$ and the K- and L-theoretic Farrell-Jones conjecture is true for $\hat{\Gamma}$ by Proposition 6.1, hence they are true for $BS(1, d)$.

Corollary 6.2. *The K- and L-theoretic Farrell-Jones conjecture is true for every semi-direct product $\mathbb{Z}[\frac{1}{p}] \rtimes C$, where p is any prime number and C is any virtually cyclic group.*

In order to prove this corollary we need the following result from [2], Theorem 1.7 which generalizes [12], Proposition 2.2.

Lemma 6.2. *Let $1 \rightarrow K \rightarrow G \xrightarrow{\psi} Q \rightarrow 1$ be an exact sequence of groups. Suppose that the group Q and for any virtually cyclic subgroup $V \subseteq Q$ the group $\psi^{-1}(V)$*

satisfies the K -theoretic Farrell-Jones conjecture. Then G satisfies the K -theoretic Farrell-Jones conjecture. The same is true for the L -theoretic Farrell-Jones conjecture.

Proof of the Corollary. The group C acts on $\mathbb{Z}[\frac{1}{p}]$ by some representation (homomorphism)

$$\varphi : C \rightarrow \text{Aut}(\mathbb{Z}[\frac{1}{p}]) = U(\mathbb{Z}[\frac{1}{p}]) = \{\pm p^{-k} \mid k \in \mathbb{Z}\}.$$

Hence there is a canonical homomorphism

$$\bar{\varphi} : \mathbb{Z}[\frac{1}{p}] \rtimes_{\varphi} C \rightarrow \mathbb{Z}[\frac{1}{p}] \rtimes U(\mathbb{Z}[\frac{1}{p}])$$

whose kernel K is a subgroup of C and therefore also virtually cyclic. Consider the following exact sequence

$$1 \longrightarrow K \longrightarrow \mathbb{Z}[\frac{1}{p}] \rtimes_{\varphi} C \xrightarrow{\bar{\varphi}} I \longrightarrow 1$$

where $I = \text{image}(\bar{\varphi})$. To prove the corollary, we want to apply Lemma 6.2 to this exact sequence. First by Theorem 1.8 from [2], the group I satisfies the K - and L -theoretic Farrell-Jones conjecture since its overgroup $\mathbb{Z}[\frac{1}{p}] \rtimes U(\mathbb{Z}[\frac{1}{p}])$ does by Proposition 6.1. Therefore to complete the proof, by Lemma 6.2, it remains to show that, for each virtually cyclic subgroup V of I , $\bar{\varphi}^{-1}(V)$ also satisfies the K - and L -theoretic Farrell-Jones conjecture. But this follows from Theorem 0.1 of [2] since $\bar{\varphi}^{-1}(V)$ is a virtually poly- \mathbb{Z} group. Note that $\bar{\varphi}^{-1}(V)$ is an extension of V by K , which are both virtually poly- \mathbb{Z} groups, hence by Lemma 4.2 (v) in [2], $\bar{\varphi}^{-1}(V)$ is a virtually poly- \mathbb{Z} group. \square

Remark 6.2. The groups $\mathbb{Z}[\frac{1}{p}] \rtimes C$ (where the image of φ is infinite) forms an interesting subclass of nearly crystallographic groups as defined in [[13], page 309]. The importance of the class of nearly crystallographic groups lies in Theorem 1.2 of [13], where it was shown that the fibered isomorphism conjecture for the stable topological pseudo-isotopy functor is true for all virtually solvable groups provided it is true for the much smaller class consisting of all nearly crystallographic groups. And the truth of the fibered isomorphism conjecture for a torsion free group G implies that $Wh(G) = 0$.

APPENDIX

Let G_d be the group $\left\{ \begin{pmatrix} d^n \cdot \frac{s_1}{s_2} & b \\ 0 & 1 \end{pmatrix} \mid s_1, s_2 \text{ are positive integers coprime to } d, b \in \mathbb{Q} \right\}$, T_d be the oriented infinite, regular, $(d + 1)$ -valent tree with edge length 1 as in

Section 2. In this appendix we explain how does the group G_d acts on the tree T_d where d is a positive integer.

If d is a prime, the action is well known as we explained in section 2. For more information, see for example [16] or [17].

We now assume d is a power of prime, hence $d = d_1^j$ for some prime d_1 . Since $G_{d_1^j}$ is a subgroup of G_{d_1} , $G_{d_1^j}$ also acts on T_{d_1} . The following definition of Stallings' folding is taken from [7].

Definition .1. *Let T be a G -tree (recall that this means, in particular, that there are no inversions). Consider two edges e_1 and e_2 in T that are incident to a common vertex v . By $\phi : e_1 \rightarrow e_2$ denote the linear homeomorphism fixing v . Then define an equivalence relation " \sim " on T as the smallest equivalence relation such that:*

- (i) $x \sim \phi(x)$, for all $x \in e_1$, and
- (ii) if $x \sim y$ and $g \in G$ then $g(x) \sim g(y)$.

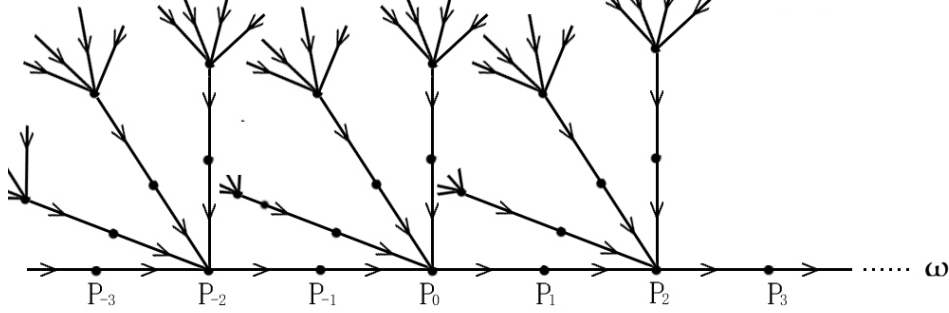
The quotient space T/\sim is a simplicial tree with a natural simplicial action of G . It might happen that G acts with inversions on T/\sim , in which case we introduce a new equivalence class of vertices to obtain a G -tree. Call the quotient map $T \rightarrow T/\sim$ a fold.

The key thing is that after the folding, G has an induced action on the new tree.

Now we take the tree to be T_2 which is the oriented infinite regular 3-valent tree we defined before; c.f., figure 1 in section 2. We first explain how does the action of G_4 on T_2 induce an action on the tree T_4 . The folding map we are going to use is $\phi : P_0P_1 \rightarrow Q_1P_1$ which fixes P_1 . Note G_4 is generated by matrices of the following three forms:

- type I: $\begin{pmatrix} 4^n & 0 \\ 0 & 1 \end{pmatrix}$;
- type II: $\begin{pmatrix} \frac{s_1}{s_2} & 0 \\ 0 & 1 \end{pmatrix}$, where s_1, s_2 are positive integers that are coprime to 2;
- type III: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, where $b \in \mathbb{Q}$

Matrices of type I act as translation on T_2 which will change the Busmann function by an even number while matrices of type II and III leave the Busmann function unchanged. Figure 3 shows the resulting tree T_2/\sim , which is homeomorphic to T_4 . If we delete all the 2-valent vertices in the resulting tree, it will be exactly T_4 ; in figure 3, for example vertices P_{2k+1} will be deleted. The group G_4 has an induced action on T_4 .

FIGURE 3. T_2 after Stallings' folding

One can generalize this to get the action of G_{2^l} on T_{2^l} from the action of G_{2^l} on T_2 by Stallings' folding. The case when d_1 is not 2 but some other prime is the same.

If d is not a power of a prime, let $d_1^{j_1} d_2^{j_2} \dots d_m^{j_m}$ be its prime factorization. Note that G_d is a subgroup of $G_{d_l^{j_l}}$ for $1 \leq l \leq m$; hence it acts on $T_{d_l^{j_l}}$, and therefore it will act on the product space $Y_d = T_{d_1^{j_1}} \times T_{d_2^{j_2}} \times \dots \times T_{d_m^{j_m}}$ diagonally. Now consider the "diagonal subspace"

$$\{y = (y_1, y_2, \dots, y_N) \in Y_d \mid f_{d_l^{j_l}}(y_l) = f_{d_{l'}^{j_{l'}}}(y_{l'}), \text{ for any } 1 \leq l \text{ and } l' \leq N\}$$

where $f_{d_l^{j_l}}$ is the Busemann function on the corresponding $T_{d_l^{j_l}}$ we defined before; see Remark 2.1. The diagonal space is an invariant subspace of G_d , hence G_d has an induced action on it. It is not too hard to show that this subspace is homeomorphic to the tree T_d we defined before; we will think of them as the same. Therefore we have an induced action of G_d on T_d . There is a natural Busemann function defined on T_d by $f_d(y) = f_{d_1^{j_1}}(y_1)$ which is the same as we defined before; see Remark 2.1.

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Tom Farrell

DEPARTMENT OF MATHEMATICS, SUNY BINGHAMTON, NY,13902,U.S.A.

E-mail address: farrell@math.binghamton.edu

Xiaolei Wu

DEPARTMENT OF MATHEMATICS, SUNY BINGHAMTON, NY,13902,U.S.A.

E-mail address: xwu@math.binghamton.edu